

ORBITAL MEASURES ON $SU(2)/SO(2)$

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ABSTRACT. We let $U = SU(2)$ and $K = SO(2)$ and denote $N_U(K)$ the normalizer of K in U . For a an element of $U \setminus N_U(K)$, we let μ_a be the normalized singular measure supported in KaK . For p a positive integer, it was proved in [4] that $\mu_a^{(p)}$, the convolution of p copies of μ_a , is absolutely continuous with respect to the Haar measure of the group U as soon as $p \geq 2$. The aim of this paper is to go a step further by proving the following two results : (i) for every a in $U \setminus N_U(K)$ and every integer $p \geq 3$, the Radon-Nikodym derivative of $\mu_a^{(p)}$ with respect to the Haar measure m_U on U , namely $d\mu_a^{(p)}/dm_U$, is in $L^2(U)$, and (ii) there exist a in $U \setminus N_U(K)$ for which $d\mu_a^{(2)}/dm_U$ is not in $L^2(U)$, hence a counter example to the dichotomy conjecture stated in [6]. Since $L^2(G) \subseteq L^1(G)$, our result gives in particular a new proof of the result in [4] when $p > 2$.

1. INTRODUCTION

Let M be a symmetric space of non compact type and G be the identity component of the isometry group of M . Let p be in M and let K be the subgroup of G fixing p that is, $K = \{g \in G \mid gp = p\}$. It is well known that G is a semisimple Lie group with trivial center and no compact factor and K is a maximal compact subgroup of G (see [3] or [7]). Moreover, the map $\zeta : G/K \rightarrow M$, defined by $\zeta(gK) = gp$ is a diffeomorphism, and if we endow G/K with the pull back of the metric of M , then ζ becomes an isometry.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra \mathfrak{g} of G . Let now $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and let U be a Lie group with Lie algebra \mathfrak{u} . Then U is a compact group and $\widetilde{M} = U/K$ is a compact symmetric space, called the dual of M .

Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} , H be an element of \mathfrak{a} and $a = \exp(iH)$. We finally define μ_a to be the normalized singular measure supported in the double coset KaK . For a positive integer p , we denote by $\mu_a^{(p)}$ the convolution of p copies of μ_a and we let

$$p_{U/K}(a) = \min \left\{ p \in \mathbb{N} \text{ such that } \mu_a^{(p)} \text{ is absolutely continuous with respect to } m_U \right\}$$

where m_U is the Haar measure of the group U .

In the case $U = SU(n)$ and $K = SO(n)$, the second author and K. Hare proved in [4] that for any point a not in the normalizer of $SO(n)$ in $SU(n)$, $p_{SU(n)/SO(n)}(a)$ is equal to the rank of the symmetric space $SU(n)/SO(n)$, that is,

$$p_{SU(n)/SO(n)}(a) = n.$$

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In [6], S. K. Gupta and K. Hare conjectured that the Radon-Nikodym derivative of $\mu_a^{(p)}$ with respect to the Haar measure m_U , which is thereafter denoted by $d\mu_a^{(p)}/dm_U$, is in $L^2(U)$ as soon as $p \geq p_{U/K}(a)$.

In this paper, we start the investigation of the L^2 -regularity of $d\mu_a^{(p)}/dm_{SU(2)}$ since it turns out that already the case of $SU(2)$ needs quite a lot of efforts.

As a first result, we give a counter example to the Gupta-Hare conjecture mentioned above in the case $U = SU(2)$. We also prove that

$$\frac{d\mu_a^{(p)}}{dm_{SU(2)}} \in L^2(SU(2))$$

as soon as $p \geq 3$ and, as a corollary, we give a new proof of the result quoted above on the absolute continuity of $\mu_a^{(p)}$ in the case $p \geq 3$ and $n = 2$.

Here is the plan of the present article. In Section 2, we say a bit more about the normalized singular measure supported in the double coset KaK and announce our main result (Theorem 1). In section 3, we present the general setting of Fourier transform of orbital measures that we make precise in the case of $SU(2)/SO(2)$ in Section 4. Sections 5 and 6 contain the two parts of the proof of our main theorem.

2. ORBITAL MEASURES AND THE MAIN THEOREM

Let K and U as above and consider the natural action of $K \times K$ on U defined as follows

$$\begin{aligned} \sigma : (K \times K) \times U &\longrightarrow U \\ ((k_1, k_2), a) &\longmapsto k_1 a k_2. \end{aligned}$$

The orbit of a point $a \in U$ under the action σ is the double coset space KaK . Since $U = KAK$, where $A = \exp(i\mathfrak{a})$ (see [9], p. 458), we will assume in all what follows that $a = \exp(iH)$, with H in \mathfrak{a} .

Each orbit is equipped with a unique $K \times K$ -invariant (K -bi-invariant) measure μ_a , defined as follows : for any f in $C(U)$, the set of continuous functions on U , we put

$$\langle \mu_a, f \rangle = \int_U f(x) d\mu_a(x) = \int_K \int_K f(k_1 a k_2) dm_K(k_1) dm_K(k_2).$$

The support of the measure μ_a , denoted by $\text{supp}(\mu_a)$, is KaK . Since $\text{supp}(\mu_a) = KaK$ has an empty interior in U , the measure μ_a is singular with respect to the Haar measure m_U of the group U . Clearly, μ_a is continuous if and if only a is not in the normalizer of K in U .

We recall that the convolution of two measures μ and ν on U , denoted by $\mu * \nu$, is defined, for any $f \in C(U)$, by

$$\langle \mu * \nu, f \rangle = \int_U f(x) d(\mu * \nu) = \int_U \int_U f(xy) d\mu(x) d\nu(y).$$

Denote the convolution of p copies of a given measure μ , namely $\mu * \mu * \dots * \mu$ (with $p - 1$ signs $*$) by $\mu^{(p)}$. Then for $f \in C(U)$,

$$\langle \mu_a^{(p)}, f \rangle = \int_U \dots \int_U f(g_1 \dots g_p) d\mu_a(g_1) \dots d\mu_a(g_p).$$

Since $\text{supp}(\mu_a) = KaK$, we have

$$\text{supp}(\mu_a^{(p)}) = (KaK)^p.$$

Specializing what we mentioned above, it was proved in [4] that $\mu_a^{(p)}$ is absolutely continuous with respect to the Haar measure if and only if $p \geq 2$. Then the Radon-Nikodym Theorem asserts the existence of a function $f_{a,p} \in L^1(U, dm_U)$ such that for $p \geq 2$,

$$d\mu_a^{(p)} = f_{a,p} dm_U.$$

Here, we shall prove that $d\mu_a^{(p)}/dm_U$ is actually in $L^2(U)$ provided $p \geq 3$. More precisely, we prove the following result, which is our main result.

Theorem 1. *Let $U = SU(2)$, $K = SO(2)$ and denote $N_U(K)$ the normalizer of K in U . For a an element of $U \setminus N_U(K)$, let μ_a denote the normalized singular measure supported in KaK . Then*

- (i) *for $p \geq 3$, the measure $\mu_a^{(p)}$ is absolutely continuous with respect to the Haar measure on U and $d\mu_a^{(p)}/dm_U$ is in $L^2(U)$,*
- (ii) *there exist an element $a \in U \setminus N_U(K)$ for which $d\mu_a^{(2)}/dm_U$ is not in $L^2(U)$.*

Sections 5 contains the proof of point (i) of Theorem 1 while Section 6 contains the one of (ii) in a stronger form (much more than one element a will be exhibited for which $d\mu_a^{(2)}/dm_U$ is not in $L^2(U)$).

This work is motivated by the article of the second-named author (joint with K. Hare) in [5] in which it was proved that if G is a compact, simple Lie group and a is not in the center of G , then μ_a , the measure supported on the conjugacy class of a , satisfies the following dichotomy: for each natural number p ,

$$\text{either } \mu_a^{(p)} \text{ is singular} \quad \text{or} \quad \mu_a^{(p)} \in L^2(G)$$

In other words,

$$(1) \quad \mu_a^{(p)} \in L^1(G) \quad \text{if and only if} \quad \mu_a^{(p)} \in L^2(G).$$

Since compact simple Lie groups can be seen as compact symmetric spaces by identifying G with $(G \times G)/\Delta$, where Δ is the diagonal of $G \times G$, these authors further conjectured in [6] that the dichotomy above holds for any compact symmetric space. More precisely, they conjectured that if U/K is a compact symmetric space and a is not in the normalizer of K in U and μ_a is the normalized singular measure supported in KaK , then (1) holds.

Theorem 1 shows that the dichotomy conjecture is false on the compact symmetric space $SU(2)/SO(2)$.

3. FOURIER TRANSFORM OF ORBITAL MEASURES

In this section, U/K will denote an arbitrary compact symmetric space. For each irreducible representation $\pi : U \rightarrow GL(E_\pi)$ of U , we fix a U -invariant inner product in E_π , which we denote for simplicity by (\cdot, \cdot) and denote by $\|\cdot\|$ the corresponding norm. The set of equivalence classes of irreducible unitary representations of U will be denoted by \hat{U} . An irreducible unitary representation (π, E_π) is called *of class one*, or *spherical*, if $\dim E_\pi^K = 1$, where

$$E_\pi^K = \{X \text{ in } E_\pi \mid \pi(k)X = X \text{ for all } k \text{ in } K\}.$$

The Fourier transform of μ_a at an irreducible unitary representation $\pi : U \rightarrow GL(E_\pi)$, denoted by $\widehat{\mu_a}(\pi)$, is an element of $\text{End}(E_\pi)$ given by (see [8], p. 77),

$$\widehat{\mu_a}(\pi)(X) := \int_U \pi(g^{-1}) X d\mu_a(g).$$

Note first that the U -invariant inner product (\cdot, \cdot) in E_π induces a Hilbert structure on $\text{End}(E_\pi)$ defined as follows: we define the Hilbert-Schmidt inner product of two elements $S, T \in \text{End}(E_\pi)$, denoted $(S, T)_{\text{HS}}$, by the formula

$$(S, T)_{\text{HS}} = \sum_{i=1}^{\dim E_\pi} (S e_i, T e_i),$$

where $\{e_1, \dots, e_{\dim E_\pi}\}$ is an orthonormal basis of E_π . It can be proved that the Hilbert-Schmidt inner product is independent of the choice of the orthonormal basis of E_π . We denote the corresponding norm by $\|\cdot\|_{\text{HS}}$, that is, for $T \in \text{End}(E_\pi)$, we put

$$\|T\|_{\text{HS}}^2 = (T, T)_{\text{HS}} = \sum_{i=1}^{\dim E_\pi} \|T e_i\|_\pi^2$$

and

$$\left\| \widehat{\mu_a^{(p)}} \right\|_2^2 = \sum_{[\pi] \in \widehat{U}} (\dim E_\pi) \left\| \widehat{\mu^{(p)}}(\pi) \right\|_{\text{HS}}^2.$$

For more details on the material of this section see [1], or [8].

The aim of this section is to prove the following result.

Proposition 2. *Let $p \geq 2$. Then*

$$\left\| \widehat{\mu_a^{(p)}} \right\|_2^2 = \sum_{\substack{[\pi] \in \widehat{U} \\ \pi \text{ spherical}}} (\dim E_\pi) |(\pi(a) X_\pi, X_\pi)|^{2p}.$$

where X_π a basis for E_π^K with $\|X_\pi\| = 1$.

In order to prove the proposition we first need four preparatory Lemmas. Let

$$C^\#(U) = \{f \text{ in } C(U) \mid f(k_1 g k_2) = f(g) \text{ for all } k_1, k_2 \text{ in } K \text{ and } g \text{ in } U\}.$$

The pair (U, K) is said to be a *Gelfand pair* if the algebra $C^\#(U)$, under convolution, is abelian.

Lemma 3. *The space E_π^K is at most one-dimensional, or equivalently,*

$$\dim E_\pi^K = 0 \quad \text{or} \quad \dim E_\pi^K = 1.$$

Proof. Since U/K is a symmetric space, the pair (U, K) is a Gelfand pair (see for example [11], Corollary 8.1.4). The Lemma follows from Proposition 6.3.1 of [2] which says that (U, K) is a Gelfand pair if and only if the space E_π^K of K -fixed vectors is at most one-dimensional. \square

Lemma 4. *If $\dim E_\pi^K = 0$, then $\widehat{\mu_a}(\pi) = 0$.*

Proof. From the definition of μ_a , we get the equalities

$$\begin{aligned} \widehat{\mu_a}(\pi)(X) &= \int_U \pi(g^{-1})(X) d\mu_a(g) \\ &= \int_{K \times K} \pi((k_1 a k_2)^{-1})(X) dk_1 dk_2 \\ &= \int_{K \times K} \pi(k_2^{-1}) \pi(a^{-1}) \pi(k_1^{-1})(X) dk_1 dk_2. \end{aligned}$$

Then we obtain

$$\begin{aligned}
 \pi(k) \widehat{\mu}_a(\pi)(X) &= \pi(k) \int_U \pi(g^{-1})(X) d\mu_a(g) \\
 &= \pi(k) \int_{K \times K} \pi(k_2^{-1}) \pi(a^{-1}) \pi(k_1^{-1})(X) dk_1 dk_2 \\
 &= \pi(k) \int_K \pi(k_2^{-1}) \pi(a^{-1}) \left(\int_K \pi(k_1^{-1})(X) dk_1 \right) dk_2 \\
 &= \int_K \pi((k_2 k^{-1})^{-1}) \pi(a^{-1}) \left(\int_K \pi(k_1^{-1})(X) dk_1 \right) dk_2 \\
 &= \int_K \pi(k_2^{-1}) \pi(a^{-1}) \left(\int_K \pi(k_1^{-1})(X) dk_1 \right) dk_2 \\
 &= \int_{K \times K} \pi(k_2^{-1}) \pi(a^{-1}) \pi(k_1^{-1})(X) dk_1 dk_2, \\
 &= \widehat{\mu}_a(\pi)(X).
 \end{aligned}$$

Therefore $\widehat{\mu}_a(\pi)(X) \in E_\pi^K$. The Lemma follows from the assumption $E_\pi^K = \{0\}$. \square

Lemma 5. *Let (π, E_π) be a spherical representation, i.e., $\dim E_\pi^K = 1$, X_π be a basis for E_π^K with $\|X_\pi\| = 1$ and let $\{X_1, \dots, X_{d_\pi}\}$ be an orthonormal basis of E_π with $X_1 = X_\pi$. Then for every positive integer k ,*

$$\left(\widehat{\mu}_a^k(\pi) X_i, X_j \right) = 0 \text{ for all pairs } (i, j) \neq (1, 1).$$

Proof. From the properties of the Fourier Transform, we have

$$\widehat{\mu}_a^k(\pi) = \widehat{\mu}_a(\pi) \dots \widehat{\mu}_a(\pi) \quad (k \text{ times}).$$

As in the proof of Lemma 4, we have

$$\pi(k) \widehat{\mu}_a(\pi)(X) = \widehat{\mu}_a(\pi)(X) \text{ for all } k \text{ in } K \text{ and all } X \in E_\pi,$$

i.e.,

$$\widehat{\mu}_a(\pi)(X) \in E_\pi^K = \text{span}\{X_1\} \text{ for all } X \text{ in } E_\pi.$$

Thus

$$\widehat{\mu}_a^k(\pi)(X) \in E_\pi^K = \text{span}\{X_1\} \text{ for all } X \text{ in } E_\pi.$$

Let k in K , X in E_π and let $P(X) = \int_K \pi(h^{-1})(X) dh$. Then

$$\begin{aligned}
 \pi(k) P(X) &= \int_K \pi(k) \pi(h^{-1})(X) dh \\
 &= \int_K \pi((hk^{-1})^{-1})(X) dh \\
 &= P(X).
 \end{aligned}$$

Hence

$$P(X) \in E_\pi^K \text{ for all } X \in E_\pi.$$

From $P(X_i) = 0$ for $i \geq 2$, we deduce that

$$\begin{aligned}\widehat{\mu}_a^k(\pi)(X_i) &= \widehat{\mu}_a^{k-1}(\pi)(\widehat{\mu}_a(\pi)(X_i)) \\ &= \widehat{\mu}_a^{k-1}(\pi)\left(\int_K \pi(k_2^{-1})\pi(a^{-1})P(X_i)dk_2\right) \\ &= 0\end{aligned}$$

since $P(X_i) = 0$ for $i \geq 2$. Therefore we obtain

$$\left(\widehat{\mu}_a^k(\pi)X_i, X_j\right) = 0 \text{ for all } i \text{ and for all } j \neq 1.$$

Hence

$$\left(\widehat{\mu}_a^k(\pi)X_i, X_j\right) = 0 \text{ for all } (i, j) \neq (1, 1).$$

□

Here is our final preparatory lemma.

Lemma 6. *With the preceding notation, one has*

$$\left(\widehat{\mu}_a^p(\pi)X_1, X_1\right) = \left(\pi(a^{-1})X_1, X_1\right)^p.$$

Proof. From the proof of Lemma 4, we have

$$\begin{aligned}\widehat{\mu}_a(\pi)X_1 &= \int_K \pi(k_2^{-1})\pi(a^{-1})\left(\int_K \pi(k_1^{-1})(X_1)dk_1\right)dk_2 \\ &= \int_K \pi(k_2^{-1})\pi(a^{-1})P(X_1)dk_2 \\ &= \int_K \pi(k_2^{-1})\pi(a^{-1})X_1dk_2\end{aligned}$$

since $P(X_1) = X_1$ as $X_1 \in E_\pi^K$. It follows

$$\widehat{\mu}_a(\pi)X_1 = P(\pi(a^{-1})X_1).$$

Hence

$$\begin{aligned}(2) \quad \left(\widehat{\mu}_a(\pi)X_1, X_1\right) &= (P(\pi(a^{-1})X_1), X_1) \\ &= (\pi(a^{-1})X_1, P^*(X_1)) \\ &= (\pi(a^{-1})X_1, X_1),\end{aligned}$$

the last equality following from the fact that P is the projection on $\text{span}\{X_1\}$, hence $P^* = P$.

By Lemma 5, in the orthonormal basis $\{X_1, \dots, X_{d_\pi}\}$ of E_π , we have

$$\left(\widehat{\mu}_a^p(\pi)X_i, X_j\right) = 0 \text{ for all } (i, j) \neq (1, 1),$$

hence the matrix associated to the endomorphism $\widehat{\mu}_a^p(\pi)$, denoted by $M\left(\widehat{\mu}_a^p(\pi)\right)$, is given by

$$M\left(\widehat{\mu}_a^p(\pi)\right) = \begin{pmatrix} \left(\widehat{\mu}_a^p(\pi)X_1, X_1\right) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since

$$\widehat{\mu_a^p}(\pi) = \widehat{\mu_a}(\pi) \dots \widehat{\mu_a}(\pi) \quad (p \text{ times}),$$

we infer

$$M\left(\widehat{\mu_a^p}(\pi)\right) = \begin{pmatrix} (\widehat{\mu_a}(\pi) X_1, X_1)^p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} \left(\widehat{\mu_a^p}(\pi) X_1, X_1\right) &= (\widehat{\mu_a}(\pi) X_1, X_1)^p \\ &= (\pi(a^{-1}) X_1, X_1)^p \end{aligned}$$

using (2), which proves our claim. \square

We are now ready to give the proof of our main proposition of this section.

Proof of Proposition 2. We use the notations of Lemma 5. One has

$$\begin{aligned} \left\|\widehat{\mu^{(p)}}(\pi)\right\|_{\text{HS}}^2 &= \sum_{i=1}^{\dim E_\pi} \left\|\widehat{\mu_a^p}(\pi) X_i\right\|^2 \\ &= \sum_{i=1}^{\dim E_\pi} \sum_{j=1}^{\dim E_\pi} \left|(\widehat{\mu_a^p}(\pi) X_i, X_j)\right|^2 \\ &= \left|(\widehat{\mu_a^p}(\pi) X_1, X_1)\right|^2 \\ &= \left|(\pi(a^{-1}) X_1, X_1)\right|^{2p} \end{aligned}$$

by Lemma 5 and then Lemma 6. Using the fact that π is unitary, we deduce that

$$\left\|\widehat{\mu^{(p)}}(\pi)\right\|_{\text{HS}}^2 = |(\pi(a) X_1, X_1)|^{2p}$$

Hence

$$\begin{aligned} \left\|\widehat{\mu_a^{(p)}}\right\|_2^2 &= \sum_{\substack{[\pi] \in \widehat{U} \\ \pi \text{ spherical}}} (\dim E_\pi) \left\|\widehat{\mu^{(p)}}(\pi)\right\|_{\text{HS}}^2 \\ &= \sum_{\substack{[\pi] \in \widehat{U} \\ \pi \text{ spherical}}} (\dim E_\pi) |(\pi(a) X_\pi, X_\pi)|^{2p}, \end{aligned}$$

as announced in the statement of the Proposition. \square

4. THE CASE OF $SU(2)/SO(2)$

Let

$$E_n = \text{span} \{z_1^k z_2^{n-k} \mid 0 \leq k \leq n\}$$

the complex vector space of homogeneous polynomials in two variables. There is a natural action of $SU(2)$ on E_n as follows:

$$\begin{aligned} \pi_n : \quad SU(2) &\longrightarrow GL(E_n) \\ A &\longmapsto \pi_n(A) \end{aligned}$$

where

$$\pi_n(A)(P(z_1, z_2)) = P((z_1, z_2)A).$$

It can be shown that π_n is irreducible and every irreducible representation of $SU(2)$ is of that form (see [11], Proposition 5.7.5). Therefore $\widehat{SU(2)} \simeq \mathbb{N}$ and the dimension of the representation corresponding to n is $n+1$.

Consider the inner product in E_n defined as follows:

$$\left(\sum_{i=0}^n a_i z_1^i z_2^{n-i}, \sum_{j=0}^n b_j z_1^j z_2^{n-j} \right) = \sum_{k=0}^n k! (n-k)! a_k \bar{b}_k.$$

With this product, the representation $\pi_n : SU(2) \rightarrow GL(E_n)$ is unitary. Denote by $\|\cdot\|$ the corresponding norm.

The vector $X_{\pi_{2n}} = (z_1^2 + z_2^2)^n$ is invariant under the action of $SO(2)$. Hence (π_{2n}, E_{2n}) is of class one. Conversely, every class one irreducible representation of $SU(2)$ is of this form. For simplicity, we put $X_{\pi_{2n}} = X_\pi$.

From

$$X_\pi = (z_1^2 + z_2^2)^n = \sum_{k=0}^n \binom{n}{k} z_1^{2k} z_2^{2(n-k)},$$

we get

$$\begin{aligned} \|X_\pi\|^2 &= \left(\sum_{k=0}^n \binom{n}{k} z_1^{2k} z_2^{2(n-k)}, \sum_{k=0}^n \binom{n}{k} z_1^{2k} z_2^{2(n-k)} \right) \\ &= \sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2. \end{aligned}$$

Let

$$\tilde{X}_\pi = \frac{X_\pi}{\|X_\pi\|} \quad \text{and} \quad a_\vartheta = \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{pmatrix}$$

where a is not in the normalizer of $SO(2)$ in $SU(2)$.

From now on, we shall use the notation already introduced $U = SU(2)$ and $K = SO(2)$. The proof of the following proposition is straightforward.

Proposition 7. *Let $A = \exp(i\mathfrak{a})$, $g = k_1 a k_2 \in U$, $k_1, k_2 \in K$, $a \in A$, and denote by $N_A(K)$ the normalizer of K in A . Then*

- (i) $g \in N_U(K)$ if and only if $a \in N_A(K)$,
- (ii) the normalizer $N_A(K)$ satisfies

$$N_A(K) = \left\{ a_\vartheta = \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{pmatrix} \mid e^{4i\vartheta} = 1 \right\}.$$

Therefore the element $a_{\pi/2} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ is clearly in the normalizer of K in U and for all $\vartheta \in (0, \frac{\pi}{2})$, $a_\vartheta \notin N_A(K)$. In the light of that fact, in all what follows, we will assume that $\vartheta \in (0, \frac{\pi}{2})$.

Put

$$\varphi_{2n}(a_\vartheta) := (\pi_{2n}(a_\vartheta) \tilde{X}_\pi, \tilde{X}_\pi).$$

Since

$$\begin{aligned} \left(\pi_{2n}(a_\vartheta) \tilde{X}_\pi \right) (z) &= \frac{1}{\|X_\pi\|} \sum_{k=0}^n \binom{n}{k} \pi_{2n}(a_\vartheta) (z_1^{2k} z_2^{2n-2k}) \\ &= \frac{1}{\|X_\pi\|} \sum_{k=0}^n \binom{n}{k} e^{i\vartheta(4k-2n)} z_1^{2k} z_2^{2n-2k}, \end{aligned}$$

we deduce that

$$\begin{aligned} \varphi_{2n}(a_\vartheta) &= \frac{1}{\|X_\pi\|^2} \left(\sum_{k=0}^n \binom{n}{k} e^{i\vartheta(4k-2n)} z_1^{2k} z_2^{2n-2k}, \sum_{k=0}^n \binom{n}{k} z_1^{2k} z_2^{2(n-k)} \right) \\ &= \frac{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2 e^{i\vartheta(4k-2n)}}{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2}. \end{aligned}$$

Therefore, applying Proposition 2, we get

$$\begin{aligned} \left\| \widehat{\mu}_{a_\vartheta}^{(p)} \right\|_2^2 &= \sum_{n=1}^{\infty} (2n+1) |\varphi_{2n}(a_\vartheta)|^{2p} \\ &= \sum_{n=1}^{\infty} (2n+1) \left| \frac{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2 e^{i\vartheta(4k-2n)}}{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2} \right|^{2p} \\ (3) \quad &= \sum_{n=1}^{\infty} (2n+1) \left| \frac{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2 e^{4ik\vartheta}}{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2} \right|^{2p}. \end{aligned}$$

5. PROOF OF THE MAIN THEOREM (i)

To study the convergence of the series (3), we first need the following easy lemma, which will elucidate the behavior of its denominator.

Lemma 8. *One has*

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n.$$

Proof. We use the method of generating functions. The generating function of the sequence $\binom{2n}{n}$ is the function $\sqrt{1-4x}$, i.e. (say for a real number x such that $|x| < 1/4$)

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

Hence, by squaring,

$$\frac{1}{1-4x} = \left(\sum_{k=0}^{\infty} \binom{2k}{k} x^k \right) \left(\sum_{l=0}^{\infty} \binom{2l}{l} x^l \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \right) x^n.$$

The Lemma follows from the identity

$$\frac{1}{1-4x} = \sum_{n=0}^{\infty} 4^n x^n.$$

□

To study the numerator appearing in the terms of the series (3), we introduce for a real number ϑ (non-zero modulo $\pi/2$) and any integer $n \geq 1$, the sum

$$(4) \quad t_n(\vartheta) = \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)$$

and investigate this quantity. Note first that t_n is periodic with period $\pi/2$.

Lemma 9. *Let ϑ be a non-zero (modulo $\pi/2$) real number. There is a positive real number $C(\vartheta)$ such that for any positive integer n , one has*

$$|t_n(\vartheta)| \leq C(\vartheta) \frac{4^n}{\sqrt{n}}.$$

Proof. We distinguish two cases depending on the parity of n .

Case a) : Let us assume first that n is odd. The sum

$$t_n(\vartheta) = \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)$$

can be written as $t_n(\vartheta) = t_n^{(1)}(\vartheta) + t_n^{(2)}(\vartheta)$, where

$$t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n-1)/2} \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)$$

and

$$t_n^{(2)}(\vartheta) = \sum_{k=(n+1)/2}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta).$$

By making the change of variable $j = n - k$ in $t_n^{(2)}(\vartheta)$, we get

$$t_n^{(2)}(\vartheta) = \exp(4in\vartheta) t_n^{(1)}(-\vartheta)$$

which yields

$$(5) \quad t_n(\vartheta) = t_n^{(1)}(\vartheta) + \exp(4in\vartheta) t_n^{(1)}(-\vartheta).$$

We therefore restrict the study of $t_n(\vartheta)$ to the one of $t_n^{(1)}(\vartheta)$. Put

$$t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n-1)/2} v_k \exp(4ik\vartheta)$$

where

$$v_k = \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Put $u_{-1}(\vartheta) = 0$ and let $u_k(\vartheta) = \sum_{j=0}^k \exp(4ij\vartheta)$ for $k \geq 0$. Then

$$u_k(\vartheta) = \frac{\exp(4i(k+1)\vartheta) - 1}{\exp(4i\vartheta) - 1} = \left(\frac{\sin(2(k+1)\vartheta)}{\sin 2\vartheta} \right) \exp(2ik\vartheta)$$

from which we get

$$|u_k(\vartheta)| = \left| \frac{\sin(2(k+1)\vartheta)}{\sin 2\vartheta} \right| \leq \frac{1}{|\sin 2\vartheta|} = c_0(\vartheta).$$

Abel's transformation gives

$$\begin{aligned} t_n^{(1)}(\vartheta) &= \sum_{k=0}^{(n-1)/2} v_k(u_k(\vartheta) - u_{k-1}(\vartheta)) \\ &= \sum_{k=0}^{(n-1)/2} v_k u_k(\vartheta) - \sum_{k=-1}^{(n-1)/2-1} v_{k+1} u_k(\vartheta) \\ &= \sum_{k=0}^{(n-1)/2-1} (v_k - v_{k+1}) u_k(\vartheta) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2}. \end{aligned}$$

We now observe that the sequence $(v_k)_{k \geq 0}$ is decreasing for $k < n/2$. Indeed, it is immediate to compute that

$$v_k - v_{k+1} = 2 \binom{2k}{k} \binom{2(n-k-1)}{n-k-1} \left(\frac{n-2k-1}{(n-k)(k+1)} \right) > 0.$$

Hence

$$\begin{aligned} |t_n^{(1)}(\vartheta)| &\leq \sum_{k=0}^{(n-1)/2-1} (v_k - v_{k+1}) |u_k(\vartheta)| + |u_{(n-1)/2}(\vartheta) v_{(n-1)/2}| \\ &\leq c_0(\vartheta) \left(\sum_{k=0}^{(n-1)/2-1} (v_k - v_{k+1}) + v_{(n-1)/2} \right) \\ &= c_0(\vartheta) \binom{2n}{n} \\ &\sim c_0(\vartheta) \frac{4^n}{\sqrt{\pi n}} \end{aligned}$$

by Stirling's formula. Thus there is a constant $c_1(\vartheta)$ such that, for all odd integers n ,

$$|t_n^{(1)}(\vartheta)| \leq c_1(\vartheta) \frac{4^n}{\sqrt{n}}.$$

It follows, by (5), that

$$\begin{aligned} |t_n(\vartheta)| &= |t_n^{(1)}(\vartheta) + \exp(4in\vartheta) t_n^{(1)}(-\vartheta)| \\ &\leq |t_n^{(1)}(\vartheta)| + |t_n^{(1)}(-\vartheta)| \\ &\leq (c_1(\vartheta) + c_1(-\vartheta)) \frac{4^n}{\sqrt{n}}. \end{aligned}$$

Hence the Lemma with $C(\vartheta) = c_1(\vartheta) + c_1(-\vartheta)$ in the case n is odd.

Case b) : Suppose now that n is even, and put

$$t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n/2)-1} \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)$$

and

$$t_n^{(2)}(\vartheta) = \sum_{k=(n/2)+1}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta).$$

By making the change of variable $j = n - k$ in $t_n^{(2)}(\vartheta)$, we get

$$t_n^{(2)}(\vartheta) = \exp(4in\vartheta) t_n^{(1)}(-\vartheta)$$

which yields

$$t_n(\vartheta) = t_n^{(1)}(\vartheta) + \exp(4in\vartheta) t_n^{(1)}(-\vartheta) + \binom{n}{n/2}^2 \exp(2in\vartheta).$$

Using the same argument as above and the fact, due again to Stirling's formula, that

$$\binom{n}{n/2}^2 \exp(2in\vartheta) = O\left(\frac{4^n}{n}\right),$$

we deduce the Lemma. \square

We are now able to deduce the key-result of this section.

Proposition 10. *Let ϑ be a non-zero (modulo $\pi/2$) real number. If $p > 2$, then the series*

$$(6) \quad \sum_{n=1}^{\infty} (2n+1) \left| \frac{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2 \exp(4ik\vartheta)}{\sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2} \right|^{2p}$$

converges.

Proof. If we put

$$\mathcal{S}_n(\vartheta) = \sum_{k=0}^n (2k)! (2n-2k)! \binom{n}{k}^2 \exp(4ik\vartheta)$$

then the series (6) is equal to

$$\sum_{n=1}^{\infty} (2n+1) \left| \frac{\mathcal{S}_n(\vartheta)}{\mathcal{S}_n(0)} \right|^{2p}.$$

But

$$\begin{aligned} \mathcal{S}_n(\vartheta) &= \sum_{k=0}^n (2k)! (2n-2k)! \frac{n!^2}{(k!(n-k)!)^2} \exp(4ik\vartheta) \\ &= \sum_{k=0}^n n!^2 \left(\frac{(2k)!}{k!^2} \right) \left(\frac{(2n-2k)!}{(n-k)!^2} \right) \exp(4ik\vartheta) \\ &= n!^2 \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta). \end{aligned}$$

We obtain that the series (6) is

$$(7) \quad \sum_{n=1}^{\infty} (2n+1) \left| \frac{\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)}{\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k}} \right|^{2p} = \sum_{n=1}^{\infty} (2n+1) \left| \frac{\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)}{4^n} \right|^{2p}$$

using Lemma 8. Lemma 9 then implies that

$$(2n+1) \left| \frac{\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta)}{4^n} \right|^{2p} \leq (2n+1) \left| \frac{C(\vartheta)}{\sqrt{n}} \right|^{2p} \ll_{\vartheta} \frac{1}{n^{p-1}}.$$

Therefore the series in (6) converges as soon as $p > 2$. \square

The proof of Theorem 1 (i) follows now easily.

Proof of the Theorem 1 (i). Combining Proposition 2, equation (3) and Proposition 10, we get

$$(8) \quad \left\| \widehat{\mu_a^{(p)}} \right\|_2^2 = \sum_{[\pi] \in \widehat{U}} (\dim E_{\pi}) \left\| \widehat{\mu_a^{(p)}}(\pi) \right\|_{\text{HS}}^2 < \infty.$$

The Plancherel isomorphism Theorem (see [8], Theorem 28.43) guarantees the existence of a function $f_{a,p} \in L^2(U)$ such that $\widehat{f_{a,p}} = \widehat{\mu_a^{(p)}}$. Hence $\mu_a^{(p)} = f_{a,p} dm_U$. \square

6. PROOF OF THE MAIN THEOREM (ii) :

A COUNTER-EXAMPLE TO THE DICHOTOMY CONJECTURE

In this section we will study the behavior of $t_n^{(1)}(\vartheta)$ introduced in the preceding section and find a lower bound for $t_n(\vartheta)$, valid for a dense enough set of indices. In this respect, we need several preliminary results (having a number-theoretical flavour) that we establish now.

First we recall the notion of *lower density* of a set of positive integers. The lower density of a set A of integers is by definition

$$\underline{d}A = \liminf_{n \rightarrow +\infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

If in this definition, we can replace the \liminf by a simple \lim then we simply speak of a *density*.

Finally, if $u \in \mathbb{R}$, we shall also denote $A + u = \{a + u \text{ for } a \in A\}$, a translate of A and $u \cdot A = \{ua \text{ for } a \in A\}$, a dilate of A .

The following lemma is crucial for our purpose.

Lemma 11. *Let A be a set of integers such that $\underline{d}A > 0$, then the series*

$$\sum_{a \in A} \frac{1}{a}$$

diverges.

Proof. Write $n_0 = 0$ and $\alpha = \underline{d}A > 0$. By definition, there is an integer n_1 such that (for instance)

$$|A \cap \{1, \dots, n_1\}| > \left\lceil \frac{\alpha n_1}{2} \right\rceil + 1$$

holds. More generally, we may construct a sequence of integer (n_i) such that

$$\begin{aligned} |A \cap \{n_{i-1} + 1, \dots, n_i\}| &= |A \cap \{1, \dots, n_i\}| - |A \cap \{1, \dots, n_{i-1}\}| \\ &> \frac{2\alpha n_i}{3} - n_{i-1} \\ &> \left\lfloor \frac{\alpha n_i}{2} \right\rfloor + 1. \end{aligned}$$

But then

$$\begin{aligned} \sum_{a \in A \cap \{n_{i-1} + 1, \dots, n_i\}} \frac{1}{a} &\geq \sum_{a = n_i - \lfloor \alpha n_i / 2 \rfloor}^{n_i} \frac{1}{a} \\ &> \int_{n_i - \lfloor \alpha n_i / 2 \rfloor}^{n_i + 1} \frac{dx}{x} \\ &= \log \left(\frac{n_i + 1}{n_i - \lfloor \alpha n_i / 2 \rfloor} \right) \\ &\sim -\log(1 - \alpha/2) > 0 \end{aligned}$$

when i tends to infinity. Summing these contributions, we get as i tends to $+\infty$,

$$\sum_{a \in A, a \leq n_i} \frac{1}{a} \gtrsim -i \log(1 - \alpha/2)$$

and the series thus diverges. \square

We shall also need the following definition. For a $\omega \in (0, \pi)$ and $c > 1/2$, we define $\mathcal{E}_{\omega, c}$ as the set of integers n such that

$$\sin \omega \sin n\omega \geq c.$$

Let now

$$\mathcal{M} = \left\{ \omega \in \mathbb{R} \text{ such that there exists } c > \frac{1}{2} \text{ such that } \underline{\mathcal{E}}_{\omega, c} > 0 \right\}.$$

Note immediately that \mathcal{M} is not empty, since for instance one can easily see that $\pi/3$ or $\pi/4$ are in \mathcal{M} . In fact, it is much bigger as shown by the following lemma for which we recall the following definition : a sequence of real numbers (x_n) is said to be uniformly distributed modulo 1 if for any $0 \leq a < b \leq 1$,

$$\lim_{n \rightarrow +\infty} \frac{|\{1 \leq m \leq n \text{ such that } \{x_m\} \in (a, b)\}|}{n} = b - a,$$

where the notation $\{x_n\}$ stands for the fractional part of x_n .

Lemma 12. *One has*

$$\mathcal{M} = \left(\frac{\pi}{6}, \frac{5\pi}{6} \right).$$

Proof. Let ω be in $(\pi/6, 5\pi/6)$ (this implies $\sin \omega > 1/2$). We consider several distinct cases.

Case a) Suppose first that ω/π is an irrational number. Then the sequence $n\omega/2\pi$ is uniformly distributed modulo 1 (see for instance Example 2.1 of [10]). Thus $n\omega$ is uniformly distributed modulo 2π .

Considering the quantity

$$q_\omega = \frac{1}{2} \left(1 + \frac{1}{2 \sin \omega} \right) \in \left(\frac{1}{2 \sin \omega}, 1 \right)$$

the uniform distribution modulo 1 property gives a positive density for those n such that

$$n\omega \pmod{\pi} \in (\arcsin q_\omega, \pi - \arcsin q_\omega),$$

a given fixed interval. It follows that the set of n such that

$$\sin n\omega > q_\omega > \frac{1}{2}$$

has a positive lower density which concludes this case.

From now on, we consider the case where

$$\omega = \frac{p\pi}{q}$$

for p and q two positive coprime numbers. Two cases remain to be investigated.

Case b) q even : recall that p and q are coprime integers. But q even implies p odd and it is sufficient to notice that p must be invertible modulo $2q$. Call p^{-1} any positive integer, inverse of p modulo $2q$. Then the integers n in

$$\frac{qp^{-1}}{2} + (2q \cdot \mathbb{N})$$

(notice that this set is an arithmetic progression of reason $2q$ and thus (positive) density $1/2q$), satisfy $pn = q/2 + 2lq$ for some integer l and thus

$$\begin{aligned} \sin n\omega &= \sin \frac{np\pi}{q} \\ &= \sin \left(\frac{1}{2} + 2l \right) \pi \\ &= \sin \left(\frac{\pi}{2} + 2l\pi \right) \\ &= 1 \end{aligned}$$

and this case is solved.

Case c) q odd: we define

$$\eta_q = \begin{cases} -1 & \text{if } q \equiv 1 \pmod{4} \\ 1 & \text{if } q \equiv 3 \pmod{4} \end{cases}$$

and notice that $q + \eta_q$ is divisible by 4. Since p is invertible modulo q (call p^{-1} any positive integer, inverse of p modulo q). Then for any

$$n \in \frac{p^{-1}(q + \eta_q)}{2} + (2q \cdot \mathbb{N})$$

(again a set with positive density, namely $1/2q$), one has $np = (q + \eta_q)/2 + 2lq$ for some integer l . Thus we obtain

$$\sin n\omega = \sin \left(\frac{q + \eta_q}{2q} + 2l \right) \pi = \sin \left(\frac{\pi}{2} + \frac{\eta_q}{2q} \pi \right) = \cos \frac{\eta_q \pi}{2q} = \cos \frac{\pi}{2q}.$$

Thus the result is proved if we can prove that

$$(9) \quad \sin \frac{p\pi}{q} \cos \frac{\pi}{2q} > \frac{1}{2}.$$

For proving (9), we notice that we can replace without loss of generality p by $q - p$. Therefore in what follows we assume

$$\frac{p\pi}{q} \leq \frac{\pi}{2}.$$

We define

$$\rho = \frac{1}{2 \cos \pi/5}$$

First, if $\sin p\pi/q > \rho$, then we obtain

$$\sin\left(\frac{p\pi}{q}\right) \cos \frac{\pi}{2q} > \frac{\cos \pi/2q}{2 \cos \pi/5} \geq \frac{1}{2}$$

since $q \geq 3$ and (9) holds.

Second, if

$$\sin \frac{p\pi}{q} \leq \rho$$

then applying the Mean-value Theorem yields

$$\begin{aligned} \sin\left(\frac{p\pi}{q}\right) - \frac{1}{2} &= \sin \frac{p\pi}{q} - \sin\left(\frac{\pi}{6}\right) \\ &\geq \left(\frac{p\pi}{q} - \frac{\pi}{6}\right) \min_{t \in [\pi/6, p\pi/q]} \cos t \\ &= \left(\frac{p}{q} - \frac{1}{6}\right) \pi \cos \frac{p\pi}{q} \end{aligned}$$

since $p\pi/q \in (\pi/6, \pi/2)$. We then obtain

$$\begin{aligned} \sin\left(\frac{p\pi}{q}\right) - \frac{1}{2} &\geq \frac{6p - q}{6q} \pi \sqrt{1 - \rho^2} \\ &\geq \frac{\pi}{6q} \sqrt{1 - \rho^2}, \end{aligned}$$

the last inequality following from the fact that the integer $6p - q$ is non-zero and thus ≥ 1 . Moreover, using the classical inequality $\cos x \geq 1 - x^2/2$ valid for x positive, we infer that

$$\sin \frac{p\pi}{q} \cos \frac{\pi}{2q} \geq \left(\frac{1}{2} + \frac{\pi}{6q} \sqrt{1 - \rho^2}\right) \left(1 - \frac{\pi^2}{8q^2}\right).$$

But the right-hand side is larger than or equal to $1/2$ as soon as

$$1 > \frac{3\pi}{8q\sqrt{1 - \rho^2}} + \frac{\pi^2}{8q^2},$$

an inequality valid for any $q \geq 3$ which concludes of (9).

The lemma is proved. \square

Here is now what we can obtain.

Lemma 13. *Let ϑ be a non-zero (modulo $\pi/2$) real number of double belonging to \mathcal{M} . Let $c_\vartheta > 1/2$ be such that $\mathcal{E}_{2\vartheta, c_\vartheta}$ has a positive lower density. Then, there is a positive real number $C'(\vartheta)$ such that*

$$|t_n(\vartheta)| \geq C'(\vartheta) \frac{4^n}{\sqrt{n}}$$

holds for any large enough n in $\mathcal{E}_{2\vartheta, c_\vartheta} - 1$.

We notice that it is well possible that the range of ϑ to which this lemma is applicable could be even extended.

Proof. We again distinguish two different cases.

Case a) : we assume first that n is odd. By (5)

$$t_n(\vartheta) = t_n^{(1)}(\vartheta) + \exp(4in\vartheta) t_n^{(1)}(-\vartheta)$$

where

$$t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n-1)/2} \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta).$$

We now study the precise behavior of $t_n^{(1)}(\vartheta)$. Recall

$$t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n-1)/2} v_k \exp(4ik\vartheta)$$

where

$$v_k = \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Recall that the sequence $(v_k)_{k \geq 0}$ is decreasing for $k \leq (n-1)/2$, $u_{-1}(\vartheta) = 0$ and

$u_k(\vartheta) = \sum_{j=0}^k \exp(4ij\vartheta)$ for $k \geq 0$. We have

$$|u_k(\vartheta)| = \left| \frac{\sin(2(k+1)\vartheta)}{\sin 2\vartheta} \right| \leq \frac{1}{|\sin 2\vartheta|} = c_0(\vartheta).$$

Abel's transformation gives

$$(10) \quad t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n-1)/2-1} (v_k - v_{k+1}) u_k(\vartheta) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2}.$$

We now define the sequence of positive real numbers ($k < n/2$)

$$w_k = v_k - v_{k+1} = 2 \binom{2k}{k} \binom{2(n-k-1)}{n-k-1} \left(\frac{1}{k+1} - \frac{1}{n-k} \right).$$

We compute

$$\begin{aligned} x_k &= w_{k+1} - w_k \\ &= 4 \binom{2k}{k} \binom{2(n-k-1)}{n-k-1} \\ &\quad \times \left(\frac{2}{(k+1)(n-k-1)} - \frac{3}{(k+1)(k+2)} - \frac{3}{(n-k)(n-k-1)} \right) \\ &< 12 \binom{2k}{k} \binom{2(n-k-1)}{n-k-1} \left(\frac{1}{(k+1)(n-k-1)} - \frac{1}{(k+1)(k+2)} \right) \\ &= \frac{12}{k+1} \binom{2k}{k} \binom{2(n-k-1)}{n-k-1} \left(\frac{1}{n-k-1} - \frac{1}{k+2} \right) \\ &\leq 0 \end{aligned}$$

for $k \leq (n-3)/2$. It follows that the sequence (w_k) is decreasing up to $(n-3)/2$.

Restarting from (10), we obtain

$$\begin{aligned}
t_n^{(1)}(\vartheta) &= \sum_{k=0}^{(n-1)/2-1} w_k u_k(\vartheta) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2} \\
&= \sum_{k=0}^{(n-3)/2} w_k \frac{\exp(4i(k+1)\vartheta) - 1}{\exp(4i\vartheta) - 1} + u_{(n-1)/2}(\vartheta) v_{(n-1)/2} \\
&= \frac{1}{\exp(4i\vartheta) - 1} \sum_{k=0}^{(n-3)/2} w_k (\exp(4i(k+1)\vartheta) - 1) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2} \\
&= \alpha(\vartheta) \sum_{k=0}^{(n-3)/2} w_k \exp(4ik\vartheta) + \beta(\vartheta) \sum_{k=0}^{(n-3)/2} w_k + u_{(n-1)/2}(\vartheta) v_{(n-1)/2} \\
&= \alpha(\vartheta) \sum_{k=0}^{(n-3)/2} w_k \exp(4ik\vartheta) + \beta(\vartheta)(v_0 - v_{(n-1)/2}) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2}
\end{aligned}$$

where we define

$$\alpha(\vartheta) = \frac{\exp(4i\vartheta)}{\exp(4i\vartheta) - 1} \quad \text{and} \quad \beta(\vartheta) = -\frac{1}{\exp(4i\vartheta) - 1}.$$

We now perform a second Abel's transformation in the first term of this expression and get

$$\begin{aligned}
t_n^{(1)}(\vartheta) &= \alpha(\vartheta) \sum_{k=0}^{(n-3)/2} w_k \exp(4ik\vartheta) + \beta(\vartheta)(v_0 - v_{(n-1)/2}) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2} \\
&= \alpha(\vartheta) \left(\sum_{k=0}^{(n-3)/2-1} (w_k - w_{k+1}) u_k(\vartheta) + u_{(n-3)/2}(\vartheta) w_{(n-3)/2} \right) \\
&\quad + \beta(\vartheta)(v_0 - v_{(n-1)/2}) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2} \\
&= \alpha(\vartheta) \left(\sum_{k=0}^{(n-5)/2} |x_k| u_k(\vartheta) + u_{(n-3)/2}(\vartheta) w_{(n-3)/2} \right) \\
&\quad + \beta(\vartheta)(v_0 - v_{(n-1)/2}) + u_{(n-1)/2}(\vartheta) v_{(n-1)/2}.
\end{aligned}$$

One now checks that

$$w_{(n-3)/2}, v_{(n-1)/2} \leq C \frac{4^n}{n}$$

for some positive constant C , which in view of the upper bound for the $u_i(\vartheta)$ implies

$$t_n^{(1)}(\vartheta) = \alpha(\vartheta) \left(\sum_{k=0}^{(n-5)/2} |x_k| u_k(\vartheta) \right) + \beta(\vartheta) v_0 + O\left(\frac{4^n}{n}\right).$$

We thus deduce that

$$\begin{aligned}
 t_n(\vartheta) &= t_n^{(1)}(\vartheta) + \exp(4in\vartheta) t_n^{(1)}(-\vartheta) \\
 &= \alpha(\vartheta) \left(\sum_{k=0}^{(n-5)/2} |x_k| u_k(\vartheta) \right) + \beta(\vartheta) v_0 \\
 &\quad + \exp(4in\vartheta) \left(\alpha(-\vartheta) \left(\sum_{k=0}^{(n-5)/2} |x_k| u_k(-\vartheta) \right) + \beta(-\vartheta) v_0 \right) + O\left(\frac{4^n}{n}\right) \\
 (11) \quad &= \sum_{k=0}^{(n-5)/2} (\alpha(\vartheta) u_k(\vartheta) + \exp(4in\vartheta) \alpha(-\vartheta) u_k(-\vartheta)) |x_k| \\
 &\quad + (\beta(\vartheta) + \exp(4in\vartheta) \beta(-\vartheta)) v_0 + O\left(\frac{4^n}{n}\right).
 \end{aligned}$$

From the identities

$$\begin{aligned}
 \alpha(-\vartheta) &= -\exp(-4i\vartheta) \alpha(\vartheta) \\
 \beta(-\vartheta) &= -\exp(4i\vartheta) \beta(\vartheta) \\
 u_k(-\vartheta) &= \exp(-4ik\vartheta) u_k(\vartheta)
 \end{aligned}$$

and (11) we get

$$\begin{aligned}
 t_n(\vartheta) &= \alpha(\vartheta) \sum_{k=0}^{(n-5)/2} (1 - \exp(4i(n-k-1)\vartheta)) u_k(\vartheta) |x_k| \\
 &\quad + (1 - \exp(4i(n+1)\vartheta)) \beta(\vartheta) v_0 + O\left(\frac{4^n}{n}\right) \\
 &= u_n(\vartheta) v_0 - \exp(4i\vartheta) \sum_{k=0}^{(n-5)/2} u_{n-k-2}(\vartheta) u_k(\vartheta) |x_k| + O\left(\frac{4^n}{n}\right).
 \end{aligned}$$

Since

$$(12) \quad |u_{n-k-2}(\vartheta) u_k(\vartheta)| \leq \frac{1}{\sin^2(2\vartheta)},$$

we get

$$\begin{aligned}
\left| \exp(4i\vartheta) \sum_{k=0}^{(n-5)/2} u_{n-k-2}(\vartheta) u_k(\vartheta) x_k \right| &\leq \frac{1}{\sin^2(2\vartheta)} \sum_{k=0}^{(n-5)/2} |x_k| \\
&= \frac{1}{\sin^2(2\vartheta)} \sum_{k=0}^{(n-5)/2} (w_k - w_{k+1}) \\
&= \frac{1}{\sin^2(2\vartheta)} \left(w_0 - w_{\frac{n-3}{2}} \right) \\
&\leq \frac{1}{\sin^2(2\vartheta)} w_0 \\
&= \frac{1}{\sin^2(2\vartheta)} (v_0 - v_1) \\
&\sim \frac{1}{2 \sin^2(2\vartheta)} v_0,
\end{aligned}$$

since $v_1 \sim v_0/2$.

To summarize, in

$$t_n(\vartheta) = u_n(\vartheta) v_0 - \exp(4i\vartheta) \sum_{k=0}^{(n-5)/2} u_{n-k-2}(\vartheta) u_k(\vartheta) |x_k| + O\left(\frac{4^n}{n}\right)$$

the first term is

$$\sim \frac{\sin(2(n+1)\vartheta)}{\sin(2\vartheta)} v_0,$$

the second one is

$$\lesssim \frac{1}{2 \sin^2(2\vartheta)} v_0$$

and the third one

$$O\left(\frac{v_0}{\sqrt{n}}\right).$$

Therefore, if $n+1 \in \mathcal{E}_{2\vartheta, c_\vartheta}$ is large enough, we deduce that the sum satisfies

$$|t_n(\vartheta)| \gg_{\vartheta} \frac{4^n}{\sqrt{n}}.$$

Case b) : If n is now even, then

$$t_n(\vartheta) = t_n^{(1)}(\vartheta) + \exp(4in\vartheta) t_n^{(1)}(-\vartheta) + \left(\frac{n}{2}\right)^2 \exp(2in\vartheta)$$

where

$$t_n^{(1)}(\vartheta) = \sum_{k=0}^{(n/2)-1} \binom{2k}{k} \binom{2n-2k}{n-k} \exp(4ik\vartheta).$$

By similar computations as above, and using

$$\left(\frac{n}{2}\right)^2 \exp(2in\vartheta) = O\left(\frac{4^n}{n}\right),$$

we get

$$t_n(\vartheta) = u_n(\vartheta) v_0 - \exp(4i\vartheta) \sum_{k=0}^{(n-4)/2} u_{n-k-2}(\vartheta) u_k(\vartheta) |x_k| + O\left(\frac{4^n}{n}\right).$$

Similar arguments as above give the conclusion.

Combining Case *a*) and Case *b*), we deduce the result announced. \square

We can now conclude this section and prove our last result.

Proof of Theorem 1 (ii). Let $\vartheta \in (\pi/12, 5\pi/12)$.

On the one hand, by Proposition 7, we see that the element

$$a_\vartheta = \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{pmatrix} \notin N_U(K).$$

On the other hand, we have $2\vartheta \in \mathcal{M}$ by Lemma 12. Thus there exists $c_\vartheta > 1/2$ such that $\mathcal{E}_{2\vartheta, c_\vartheta}$ has a positive lower density. Now, the series associated to a_ϑ by (3) is, in view of (6) and (7),

$$\begin{aligned} \left\| \widehat{\mu}_{a_\vartheta}^{(2)} \right\|_2^2 &= \sum_{n=1}^{\infty} (2n+1) \left| \frac{t_n(\vartheta)}{4^n} \right|^4 \\ &\geq \sum_{n \in \mathcal{E}_{2\vartheta, c_\vartheta} - 1} (2n+1) \left| \frac{t_n(\vartheta)}{4^n} \right|^4 \\ &\geq C'(\vartheta)^4 \sum_{n \in \mathcal{E}_{2\vartheta, c_\vartheta} - 1} \frac{2n+1}{n^2} \end{aligned}$$

by Lemma 13. It follows that

$$\left\| \widehat{\mu}_{a_\vartheta}^{(2)} \right\|_2^2 \geq 2C'(\vartheta)^4 \sum_{n \in \mathcal{E}_{2\vartheta, c_\vartheta} - 1} \frac{1}{n},$$

a series which is divergent by Lemma 11 applied to the set $\mathcal{E}_{2\vartheta, c_\vartheta} - 1$ which has a positive lower density. \square

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